JOURNAL OF APPROXIMATION THEORY 57, 69-76 (1989)

# Best Approximation by Continuous $n$-Convex Functions

A. L. BROWN

*School of Mathematics, University of Newcastle upon Tyne, NE1 7RU, England*

*Communicated by Frank Deutsch*

Received October 6, 1986; revised June 22, 1987

Let $C([0, 1])$ be the space, equipped with the uniform norm, of continuous real functions on $[0, 1]$. Let $n \geqslant 2$ and let $\mathscr{C}_n$ be the set of continuous $n$-convex functions on $[0, 1]$. The methods of [1] and the results of [2] will be used to obtain a characterisation of and a partial uniqueness theorem for a best approximation to a function $\phi \in C([0, 1]) \backslash \mathscr{C}_n$ from $\mathscr{C}_n$.

Best approximation by $n$-convex functions has been considered by Zwick [7, 8]. A more general problem was considered earlier by Burchard [3], whose results have not been published in detail. The relation of the main result of this note (Theorem 3) to that of Burchard [3, Theorem 5] is considered at the end of the note. Information concerning $n$-convex functions can also be found in [4].

A function $f$ is defined to be $n$-convex if its $n$th divided differences are non-negative. However, a function $f$ is continuous and $n$-convex on $[0, 1]$ if and only if it is continuous on $[0, 1]$ and the derivative $f^{(n-2)}$ exists and is convex on the open interval $(0, 1)$; the latter conditions will be taken as a working definition of the class $\mathscr{C}_n$. If $f \in \mathscr{C}_n$ then the left and right derivatives $f_-^{(n-1)}$ and $f_+^{(n-1)}$ exist on $(0, 1)$.

Let $P_{n-1}$ denote the space of polynomials of degree at most $n-1$ and let $K_n$ denote the kernel defined by

$$K_n(s, t) = \frac{(s-t)_+^{n-1}}{(n-1)!}.$$

To each $f \in \mathscr{C}_n$ we can associate a measure $\mu = \mu_f$ defined on the open interval $(0, 1)$ by

$$\mu([t, s]) = f_+^{(n-1)}(s) - f_-^{(n-1)}(t) \qquad \text{for} \quad 0 < t \leqslant s < 1.$$

The measure $\mu$ is a positive regular Borel measure on $(0, 1)$; it is bounded if and only if $\lim_{t \to \infty} f_-^{(n-1)}(t)$ and $\lim_{t \to \infty} f_+^{(n-1)}(t)$ both exist. If $\mu$ is



0021-9045/89 $3.00

bounded then it can be regarded as a measure on $\mathbb{R}$ with support, supp $\mu$, contained in $[0, 1]$. In this case the function $f$ has a representation

$$f(s)=\sum_{k=0}^{n-1}\frac{f_+^{(k)}(0)\,s^k}{k!}+\int K_n(s,t)\,d\mu(t) \qquad \text{for} \quad 0\leqslant s\leqslant 1.$$

A similar representation, of which this is a variant and a simple consequence, is considered in [2, Theorem 2.2]. If $\lim_{t\to\infty} f_-^{(n-1)}(t)$ does not exist then there can be no such representation. However, let $\mathscr{C}'_n$ be the set of functions $f$ on $[0, 1]$ hich have representations of the form

$$f(s)=p(s)+\int K_n(s,t)\,d\mu(t) \qquad \text{for} \quad 0\leqslant s\leqslant 1,$$

for some $p\in P_{n-1}$ and some positive regular Borel measure $\mu$ on $\mathbb{R}$ with supp $\mu\subseteq[0, 1]$. Then $\mathscr{C}'_n\subseteq\mathscr{C}_n$ (cf. [2, Theorem 2.2]). The first theorem establishes that $C'_n$ is dense in $\mathscr{C}_n$. The characteristic function of an interval $[\alpha, \beta]$ is denoted $\chi_{[\alpha,\beta]}$.

THEOREM 1. *Let $f\in\mathscr{C}_n$. For each $\alpha\in(0, 1)$ and $s\in[0, 1]$ the function $K_n(s,\cdot)\chi_{[\alpha,1)}$ is $\mu$-integrable and the equation*

$$f_\alpha(s)=\sum_{k=0}^{n-1}\frac{f_-^{(k)}(\alpha)}{k!}(s-\alpha)^k+\int K_n(s,t)\,\chi_{[\alpha,1)}(t)\,d\mu(t)$$

*defines a function $f_\alpha\in\mathscr{C}'_n$. Furthermore, $f_\alpha(s)=f(s)$ for all $s\in[\alpha, 1]$ and $f=\lim_{\alpha\to 0} f_\alpha$.*

*Proof.* Let $p_\alpha\in P_{n-1}$ be the polynomial defined by

$$p_\alpha(s)=\sum_{k=0}^{n-1}\frac{f_-^{(k)}(\alpha)}{k!}(s-\alpha)^k.$$

Let $0<\alpha<\beta<1$. Then (cf. [2, Theorem 2.2]) the function $f_{\alpha\beta}$ defined by the equation

$$f_{\alpha\beta}(s)=p_\alpha(s)+\int K_n(s,t)\,\chi_{[\alpha,\beta]}(t)\,d\mu(t)$$

has the following properties:

(i) $f_{\alpha\beta}\in\mathscr{C}'_n$,

(ii) $f_{\alpha\beta}(s)=f(s)$ for all $s\in[\alpha, \beta]$,

(iii) $f_{\alpha\beta}\,|\,[0, \alpha]$ is a polynomial of degree at most $n-1$ and its first $n-1$ derivatives at $\alpha$ coincide with the corresponding right derivatives of $f$ at $\alpha$,

(iv) $f_{\alpha\beta} | [\beta, 1]$ is a polynomial of degree at most $n-1$ and its first $n-1$ derivatives at $\beta$ coincide with the corresponding left derivatives of $f$ at $\beta$.

Note that $f_{\alpha\beta}$ is determined by Conditions (ii), (iii), and (iv), and that $\alpha$ and $\beta$ enter symmetrically into these conditions, although they do not enter symmetrically in the integral representation of $f_{\alpha\beta}$.

The function $K_n(s, t)\,\chi_{[\alpha,\beta]}(t)$ is an increasing function of $\beta$ and an increasing function of $s$. For each $s \in [\alpha, 1]$

$$f_{\alpha\beta}(s) = f(s) \qquad \text{for} \quad \beta \geqslant s,$$

and so

$$f(s) - p_\alpha(s) = \lim_{\beta \to 1} \int K_n(s, t)\chi_{[\alpha,\beta]}(t)\, d\mu(t).$$

Therefore by the monotone convergence theorem $K_n(s, \cdot)\chi_{[\alpha,1)}$ is $\mu$-integrable and

$$f(s) = p_\alpha(s) + \int K_n(s, t)\chi_{[\alpha,1)}(t)\, d\mu(t) \qquad \text{for all} \quad s \in [\alpha, 1).$$

Now $f(1) = \lim_{s \to 1} f(s)$ and so, again by the monotone convergence theorem, the preceding equation holds also for $s = 1$. It remains to prove that $f = \lim_{\alpha \to 0} f_\alpha$.

From what has been proved it follows that, for $s \in [\alpha, 1]$,

$$\begin{aligned} f(s) - f_{\alpha\beta}(s) &= \int K_n(s, t)\,\chi_{(\beta,1)}(t)\, d\mu(t) \\ &\leqslant \int K_n(1, t)\,\chi_{(\beta,1)}(t)\, d\mu(t) \\ &\to 0 \qquad \text{as} \quad \beta \to 1. \end{aligned}$$

Thus $f_{\alpha\beta}$ converges uniformly to $f$ on $[\alpha, 1]$ as $\beta \to 1$. It follows by the symmetry remarked on above that $f_{\alpha\beta}$ converges uniformly to $f$ on $[0, \beta]$ as $\alpha \to 0$. Now $f_\alpha(s) = f_{\alpha\beta}(s)$ for $0 < \alpha < \beta < 1$ and $0 \leqslant s \leqslant \beta$. It follows that $f = \lim_{\alpha \to 0} f_\alpha$. The proof is complete.

Theorem 1 will be used in obtaining the characterisation of best approximations from $\mathscr{C}_n$. In [2] a result of Sattes [5] characterising best approximations by smooth functions was obtained by a duality argument and integral representations for the smooth functions. The next theorem is closely similar to [2, Theorem 1]. We will denote by $B(\phi, r)$ and $B'(\phi, r)$, respectively, the open and closed balls in $C([0, 1])$ with centre $\phi$ and radius $r$. The distance of $\phi$ from $\mathscr{C}_n$ is denoted $d(\phi, \mathscr{C}_n)$.

THEOREM 2. *Let* $\phi \in C([0, 1])\backslash\mathscr{C}_n$. *If* $f_0 \in \mathscr{C}_n$ *and* $\mu_0$ *is the measure associated with* $f_0$, *then* $f_0$ *is a best approximation to* $\phi$ *from* $\mathscr{C}_n$ *if and only if there exists a non-zero measure* $\lambda$, *with* $\operatorname{supp} \lambda \subseteq [0, 1]$, *such that*

(a) $\int p \, d\lambda = 0$ *for all* $p \in P_{n-1}$,

(b) $g(t) := \int K_n(s, t) \, d\lambda(s) \geqslant 0$ *for all* $t \in [0, 1]$,

(c) $\operatorname{supp}(\sigma\lambda)^+ \subseteq \{s : (f_0 - \phi)(s) = \sigma \|f_0 - \phi\|\}$ *for* $\sigma = 1$ *and* $\sigma = -1$,

(d) $\operatorname{supp} \mu_0 \subseteq g^{-1}(0)$.

*Proof.* In the course of the proof $\lambda$ will be regarded interchangably as a measure and as a linear functional in $C([0, 1])^*$.

It is known [7] that given $\phi \in C([0, 1])\backslash\mathscr{C}_n$ there exists $f_0$ such that

(i) $f_0 \in \mathscr{C}_n$ and $f_0$ is a best approximation to $\phi$ from $\mathscr{C}_n$.

The convex sets $B(\phi, d(\phi, \mathscr{C}_n))$ and $\mathscr{C}_n$ are disjoint and so there exists a $\lambda$ such that

(ii) $\lambda \in C([0, 1])^*\backslash\{0\}$ and $\lambda(u) \leqslant \lambda(f)$ for all $u \in B'(\phi, d(\phi, \mathscr{C}_n))$ and all $f \in \mathscr{C}_n$.

It is easily shown, using the fact that $\mathscr{C}_n$ is a cone ($\mathbb{R}^+\mathscr{C}_n = \mathscr{C}_n$) that Conditions (i) and (ii) are together equivalent to the three conditions

(iii) $\inf \lambda(\mathscr{C}_n) = 0$ (or, equivalently, $\lambda(\mathscr{C}_n)$ is bounded below),

(iv) $\lambda(f_0) = 0$, and

(v) $\lambda(f_0 - \phi) = \|\lambda\| \, \|f_0 - \phi\|$.

(This equivalence, transposed to a general normed linear space, is an amplification of the characterisation of best approximations from a convex cone due, independently, to F. Deutsch (see [6, p. 93]) and G. Sh. Rubinshtein (see [6, p. 65).) Condition (c) is equivalent to Condition (v).

The set $\lambda(\mathscr{C}_n)$ is bounded below if and only if $\lambda(\mathscr{C}'_n)$ is bounded below, and then the two sets both have infimum zero. If

$$f = p + \int K_n(\cdot, t) \, d\mu(t) \in \mathscr{C}'_n,$$

where $p \in P_{n-1}$ and $\mu$ is a positive measure $\mathbb{R}$ with $\operatorname{supp} \mu \subseteq [0, 1]$, then, by Fubini's theorem,

$$\lambda(f) = \lambda(p) + \int g(t) \, d\mu(t).$$

The function $g$ is continuous. It follows that Condition (iii) is equivalent to the two Conditions (a) and (b) together.

Let

$$f_0 = \lim_{\alpha \to 0} \left( p_\alpha + \int K_n(\cdot, t)\chi_{[\alpha, 1)}(t)\, d\mu_0(t) \right)$$

as in Theorem 1. If Condition (a) is satisfied then, again by Fubini's theorem,

$$\begin{aligned} \lambda(f_0) &= \lambda\left( \lim_{k \to \infty} \left( p_{1/k} + \int K(\cdot, t)\chi_{[1/k, 1)}(t)\, d\mu_0(t) \right)\right) \\ &= \lim_{k \to \infty} \int g(t)\chi_{[1/k, 1)}(t)\, d\mu_0(t). \end{aligned}$$

If Condition (b) is satisfied then the latter sequence is non-negative and increasing. It follows that if (a) and (b) are satisfied then Condition (iv) is equivalent to Condition (d). The proof of the theorem is complete. Conditions (i) and (ii) are together equivalent to Conditions (a), (b), (c), and (d).

It is now possible, using the results of [2], to obtain a more explicit characterisation of a best approximation $f_0$ in $\mathscr{C}_n$ to $\phi$ in $C([0, 1])\backslash\mathscr{C}_n$. The function $g$ of Theorem 2 is defined on $\mathbb{R}$. The Condition (a) is equivalent to the condition: $g(t) = 0$ for all $t \leqslant \inf(\operatorname{supp} \lambda)$. (Necessarily, by the form of $K_n$, $g(t) = 0$ for all $t \geqslant \sup(\operatorname{supp} \lambda)$.) By Condition (b) the isolated zeros of $g$ are all of even multiplicity. Condition (d) explicitly concerns the zeros of $g$. It follows from Condition (c) that $g$ belongs to the class of functions, the zeros of which were investigated in [2]. By Condition (d), if the number of zeros of $g$ on an interval $(a, b)$ is finite then the restriction of $f_0$ to $(a, b)$ is a spline with simple knots at zeros of $g$. If the restriction of $f_0$ to an open interval containing a set $I$ is a spline with simple knots then $k(f_0, I)$ will denote the number of knots of $f_0$ in the set $I$. The main result of the note can now be stated and proved.

THEOREM 3. *A function $f_0 \in \mathscr{C}_n$ is a best approximation to $\phi \in C([0, 1])\backslash\mathscr{C}_n$ from $\mathscr{C}_n$ if and only if there exist $m \geqslant n + 1$ and*

$$0 \leqslant \xi_1 < \cdots < \xi_m \leqslant 1$$

*such that*

(i) *$m - n - 1$ is even,*

(ii) *$(f_0 - \phi)(\xi_i) = (-1)^{j-1-n} \|f_0 - \phi\|$ for $j = 1, \ldots, m$,*

(iii) $f_0|(\xi_1, \xi_m)$ *is a spline of degree* $n-1$ *with simple knots and satisfies the conditions*

$$\begin{array}{lll}
(1)\ k(f_0, (\xi_1, \xi_m)) & \leqslant \frac{1}{2}(m-1-n), & \\
(2)\ k(f_0, (\xi_1, \xi_l)) & \leqslant \frac{1}{2}(l-2) & \textit{for } l=2, ..., m-1, \\
(3)\ k(f_0, [\xi_k, \xi_m)) & \leqslant \frac{1}{2}(m-k-1) & \textit{for } k=2, ..., m-1,
\end{array}$$

$$(4)\ k(f_0, [\xi_{kk}, \xi_l]) \begin{cases} \leqslant \frac{1}{2}(l-k+n-2) & \textit{if } k+n \textit{ is even and } 2 \leqslant k < l \leqslant m-1, \\ < \frac{1}{2}(l-k+n-2) & \textit{if } k+n \textit{ is odd and } 2 \leqslant k < l \leqslant m-1. \end{cases}$$

*If these conditions are satisfied then each best approximation to* $\phi$ *from* $\mathscr{C}_n$ *coincides with* $f_0$ *on the interval* $[\xi_1, \xi_m]$.

*Proof.* Suppose that $f_0$ is a best approximation to $\phi$ from $\mathscr{C}_n$ and that $f_0$ and $\lambda$ satisfy the conditions of Theorem 2. The continuous function $f_0 - \phi$ alternates between $\pm \|f_0 - \phi\|$ finitely many times on $[0, 1]$. Therefore, by (c), there exist closed subsets $D_1, ..., D_m$ of $[0, 1]$ and $\varepsilon$ equal to either 1 or $-1$ such that

$$\sup D_j < \inf D_{j+1} \qquad \text{for} \quad j = 1, ..., m-1,$$

$$\operatorname{supp}(\varepsilon\lambda)^+ = D_1 \cup D_3 \cup \cdots,$$

and

$$\operatorname{supp}(\varepsilon\lambda)^- = D_2 \cup D_4 \cup \cdots.$$

If $g$ is the function of (b) then, by (a), $m \geqslant n+1$ and $g(t) = 0$ for $t \notin (\inf(\operatorname{supp} \lambda), \sup(\operatorname{supp} \lambda))$. It follows from Condition (b), by [2, Lemma 2.3], that $\varepsilon = (-1)^n = (-1)^{m+1}$. Thus, in the notation of [2], the function $g$ belongs to the class $\mathscr{S}(n, m, D, (-1)^n)$. If the function $g$ has zeros of multiplicity $n$ other than the zero intervals $(-\infty, \inf(\operatorname{supp} \lambda)]$ and $[\sup(\operatorname{supp} \lambda), \infty)$ then by the Decomposition Theorem 1.5 of [2] we can replace $\lambda$ by a measure (also to be denoted by $\lambda$) such that all the conditions remain satisfied and, also, the corresponding function $g$ has only isolated zeros of multiplicity at most $n-1$ in the interval $(\inf(\operatorname{supp} \lambda), \sup(\operatorname{supp} \lambda))$; that is, $g$ is in the class $\mathscr{S}_0(n, m, D, (-1)^n)$. By (b) the isolated zeros of $g$ are of even multiplicities [2, Corollary 3.2]. It now follows from [2, Theorem 1.6A] that there exist $\xi_1, ..., \xi_m$ such that Conditions (i), (ii), and (iii) are satisfied.

Conversely, if $m$ and $\xi_1, ..., \xi_m$ satisfy Conditions (i), (ii), and (iii) then it follows from [2, Theorem 1.6B] that there exists a measure $\lambda$ such that

$$\operatorname{supp}((-1)^n\lambda)^+ \subseteq \{\xi_1, \xi_3, ...\},$$

$$\operatorname{supp}((-1)^n\lambda)^- \subseteq \{\xi_2, \xi_4, ...\},$$

and such that the function $g = \int K(s, \cdot)\, d\lambda(s)$ has double zeros at the knots of $f_0 | (\xi_1, \xi_m)$, is non-negative, and is zero outside $(\xi_1, \xi_m)$, but has no other zeros. Then $f_0$ and $\lambda$ satisfy Conditions (a), (b), (c), and (d) of Theorem 2 and so $f_0$ is a best approximation to $\phi$ from $\mathscr{C}_n$.

Finally, the uniqueness statement must be proved. Suppose that $f_0$ is a best approximation to $\phi \in C([0, 1]) \backslash \mathscr{C}_n$ from $\mathscr{C}_n$; that $m$ and $\xi_1, ..., \xi_m$ satisfy Conditions (i), (ii), and (iii); and that $\lambda$ is as in the previous paragraph. If $f'_0 \in \mathscr{C}_n$ is also a best approximation to $\phi$ then $f'_0$ and $\lambda$ satisfy the conditions of Theorem 2 and $f'_0 | [\xi_1, \xi_m]$ is a spline, each knot of which is a knot of $f_0 | (\xi_1, \xi_m)$. Thus $(f_0 - f'_0) | [\xi_1, \xi_m]$ is a spline, each knot of which is a knot of $f_0 | (\xi_1, \xi_m)$. The knots of $(f_0 - f'_0) | [\xi_1, \xi_m]$ therefore satisfy conditions (1)–(4). Furthermore, $\xi_1, ..., \xi_m$ are zeros of $(f_0 - f'_0) | [\xi_1, \xi_m]$. It must be shown that $(f_0 - f'_0) | [\xi_1, \xi_m] = 0$. Suppose not. Then the spline $h$ defined on $\mathbb{R}$, equal to $f_0 - f'_0$ on $(\xi_1, \xi_m)$ and with the same knots, either has no zero intervals, or one, or more than one, but is not everywhere zero on $(\xi_1, \xi_m)$. The argument will be given in detail for the third case. Suppose that $h$ has no zero interval on $[\alpha, \beta] \subseteq [\xi_1, \xi_m]$ but that $\alpha$ and $\beta$ are end points of zero intervals. Let

$$\xi_k \leqslant \alpha < \xi_{k+1} \leqslant \xi_{l-1} < \beta \leqslant \xi_l.$$

Then, by (iii) (whether $k = 1$ or $k > 1$, $l = m$ or $l < m$),

$$k(h, [\alpha, \beta]) \leqslant \tfrac{1}{2}(l - k + n - 2).$$

Now, by [2, Theorem 1.6A] or by standard results on zeros of splines and the fact that $h$ is zero at $\xi_{k+1}, ..., \xi_{l-1}$, the number $Z(h, (\alpha, \beta))$ of zeros of $h$ on $(\alpha, \beta)$ satisfies the inequalities

$$l - 1 - k \leqslant Z(h, (\alpha, \beta)) \leqslant \tfrac{1}{2}(l - k + n - 2) - 1 - n,$$

which is impossible. The other two cases lead in the same way to contradictions. This completes the proof.

We conclude with a number of observations on the result.

*Remark* 1. In the case $n = 2$ of convex functions it follows from Conditions (2) $(l = 2)$, (3) $(k = m - 1)$, and (4) $(l - k = 1)$ that $f_0$ has no knots on $(\xi_1, \xi_n)$, that is, it is linear.

*Remark* 2. Theorem 3 is essentially equivalent to that particular case of [3, Theorem 5] in which the set of functions considered is the set of $n$-convex functions. However, the equivalence is not obvious or immediate. An interval $[\xi_1, \xi_m]$ satisfying the conditions of Theorem 3 can be expressed as a union of intervals (possibly overlapping), each of which satisfies the conditions of the theorem together with the further condition that the number of knots of the functions $f_0$ is the maximal number allowed by Condition (iii)(1). This is, in effect, established by the proof of [2, Theorem 1.6B]. For such intervals the conditions of Theorem 3 reduce to those of [3, Theorem 5].

## ACKNOWLEDGMENTS

This note was written while the author was a visitor at the Mathematics Institute of the Erlangen-Nürnberg University. The author is grateful to the Institute and to the Deutscher Akademischer Austauchdienst for their hospitality and to Professor D. Zwick for drawing his attention to the problem considered.

## REFERENCES

1. A. L. BROWN, Best approximation by smooth functions and related problems, *in* "Parametric Optimization and Approximation," (F. Deutsch and B. Brosowski, Eds.), pp, 70–82, Proceedings of a Conference, Oberwolfach, October 1983, Birkhäuser, Basel, 1985.
2. A. L. BROWN, On the zeros of certain functions which have a piecewise alternating convex and concave $p$th derivative, *J. London Math. Soc. (2)* **33** (1986), 311–327.
3. H. G. BURCHARD, Extremal positive splines with applications to interpolation and approximation by generalized convex functions, *Bull. Amer. Math. Soc.* **79** (1973), 959–963.
4. A. W. ROBERTS AND D. E. VARBERG, "Convex Functions," Academic Press, New York, 1973.
5. U. SATTES, Best Chebyshev approximation by smooth functions, *in* "Quantitative Approximation" (R. A. Devore and K. Scherer, Eds.), pp. 279–289, Proceedings, International Symposium, Bonn, 1979, Academic Press, New York, 1980.
6. I. SINGER, "The Theory of Best Approximation and Funtional Analysis," CBMS 13, SIAM, Philadelphia, 1974.
7. D. ZWICK, Existence of best $n$-convex approximations, *Proc. Amer. Math. Soc.* **97** (1986), 273–276.
8. D. ZWICK, Best $n$-convex approximation: A sufficient condition, *in* "Proceedings of the A. Haar Memorial Conference, Budapest, 1985."